Answer key for Yang winter 24 midterm 1

1 (1) $m = 4, n = 3$.

(2) No. Its reduced echelon form will be

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3) The linear system $A\mathbf{x} = \mathbf{b}$ has general solutions $(x_1, x_2, x_3, x_4) = (4 - 2s, s, -1, 3)$. Any value for s provides a valid answer (for example, $s = 0$ and $\mathbf{x} = (4, 0, -1, 3)$).

2 (1) The echelon form of $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the last column does not contain a leading term, the vectors are not independent.

(2) Since the echelon matrix above has a zero row, the vectors do not span $\mathbb{R}^3$.

(3) The linear system

$$\mathbf{u}_1 x_1 + \mathbf{u}_2 x_2 + \mathbf{u}_3 x_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 2 \\ 1 & -1 & -5 & -1 \end{array}\right]$$

has echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last column does not contain a leading term, the system has at least one solution, hence the vector is in the span of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

3 (1) Not possible, any three vectors in a plane cannot be linearly independent.

(2) $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$. Other solutions are possible, but you always need at least four vectors.

4 (1) The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & 0 & -8 & -2 \\ -6 & 6 & z_1 & z_2 \end{array}\right]$$

which has echelon form

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 6 & -10 & -10 \\ 0 & 0 & z_1 - 14 & z_2 + 4 \end{array}\right]$$

If we want a unique solution, we only need $z_1 \neq 14$ (z_2 can be any number).

(2) Observation: $x_2 = s$ is a free parameter. $x_3 = -1, x_4 = 2$ are exactly the right hand sides of the last two equations. Together with $x_1 = 1 + 2s = 1 + 2x_2$, we get

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5 False. Let $\mathbf{u}_1 = (1,0,0), \mathbf{u}_2 = (0,1,0), \mathbf{u}_3 = (0,0,1)$. Then

$$\mathbf{u}_1 + 2\mathbf{u}_2 = (1,2,0), \mathbf{u}_1 + \mathbf{u}_3 = (1,0,1), \mathbf{u}_2 - \mathbf{u}_3/2 = (0,1,-1/2).$$

They do not span $\mathbb{R}^3$ since they are linearly dependent:

$$(1,2,0) - (1,0,1) - 2(0,1,-1/2) = 0,$$

and any three dependent vectors in $\mathbb{R}^3$ do not span $\mathbb{R}^3$.